Adaption for 2D Edge Elements in the Nonconforming Voxel Finite Element Method

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The nonconforming voxel finite element method (NVFEM) avoids many of the difficulties of traditional finite element mesh generation by using a nested grid of rectangular elements. It models arbitrary boundary shapes by adaptively refining the mesh at the boundaries. Here the adaption is extended to reduce errors arising from elements that are away from boundaries, but are too big to represent the field adequately. The application is to 2D edge elements for solving the time-harmonic wave equation. Results for a miter-bend waveguide problem demonstrate the effectiveness of the new algorithm.

***Index Terms*—Computational electromagnetics, finite element analysis, mesh generation.**

I. Introduction

MESH GENERATION continues to be a challenging part of finite element (FE) analysis. It can be computationally intensive and, if not carefully done, it can lead to poor element shapes and ill-conditioned matrix equations. The nonconforming voxel finite element method (NVFEM) [1][2][3][4] provides an alternative to traditional mesh generation: only rectangular elements are used, but when these straddle material boundaries they are subdivided repeatedly in order to provide a better modeling of the boundary shape. This reduces the error in the FE solution and is a form of adaption that might be called "geometric". However, error also comes from elements that do not straddle a boundary, but are simply too big to represent the field variation adequately. Reducing this error requires "field" adaption. In this paper we propose an NVFEM algorithm that combines both geometric and field adaption.

II. Adaptive Algorithm

NVFEM can be applied to vector electromagnetics using edge elements [5][6]. Here we solve the vector wave equation, $\nabla\times\nabla\times\mathbf{E}={k_0}^2\varepsilon_r\mathbf{E}$, for the 2D phasor electric field $\mathbf{E}$ at a specified free-space wavenumber, k_0, using rectangular edge elements with four unknowns, one per edge [6]. Dielectric materials may be present and on the outer boundaries the tangential part of $\mathbf{E}$ is specified (e.g., as zero, in which case the boundary is a perfect electric conductor, PEC).

The FE used is rectangular and has one unknown per edge. It enforces tangential continuity of $\mathbf{E}$ between elements. The standard basis function for an x-directed edge is uniform in x and varies linearly in y. When the element straddles a boundary, the standard basis functions are replaced by computed basis functions (CBFs) which take into account the boundary and provide substantially greater accuracy [6].

Initially the geometry is placed in a rectangular box which is subdivided, coarsely, into $M\times M$ elements. A geometric refinement, GR, is one pass through the mesh, subdividing each element that straddles a boundary into four equal elements. A field refinement, FR(P), is a similar pass, subdividing the worst P% of the elements as determined by an error indicator, discussed below.

The algorithm attempts to reduce the total error in the solution systematically in such a way that, at whatever point it is terminated, neither form of refinement has been excessive in comparison with the other. Continuing too far with GRs when the geometric error is already below the field error is inefficient because it does little to reduce the overall solution error, and vice versa.

The criterion for switching between GR and FR is based on the change in the FE functional, which is given by

$$F=\int_{\Omega}\{(\nabla\times\mathbf{E})^2-k_0^2\varepsilon_r\mathbf{E}^2\}\,\mathrm{d}\Omega \qquad (1)$$

ε_r is the dielectric constant. The adaptive algorithm is:

0. Solve the FE problem with initial $M\times M$ mesh. Set $P=25$, $Q=20$, $D=2$.
1. Repeat GR until the change in F between two successive solves is less than Q%.
2. Repeat FR(P) until the change in F between two successive solves is less than Q%.
3. $Q=Q/D$.
4. If further refinement is needed and possible, go to 1.

In principle the algorithm could run for ever. In practice step 4 has to provide for a means of stopping, e.g., when the results are accurate enough or the available computational resources are exhausted.

The values of the parameters P and Q and the divisor D at step 3 are obviously somewhat arbitrary. They have been chosen based on numerical experiments.

Of the many error indicators that have been proposed, we use an inexpensive one that measures the average electric and magnetic strength in the element, weighted by its area, A:

$$\frac{A}{4}\sum_{k=1}^{4}\left(|\mathbf{E}_k|^2+\eta^2|\mathbf{H}_k|^2\right) \qquad (2)$$

$\mathbf{E}_k$ is the electric field at the midpoint of the kth edge, $\mathbf{H}_k$ is the magnetic field at the midpoint of the kth edge and η is the intirnsic impedance in the element. The reasoning behind (2) is that large elements with large fields have the biggest contribution to F and are likely also to contribute most to the error in F.

III. Numerical Results

In this section, we evaluate the performance of the new adaptive algorithm. For this purpose, we apply the method to the 90°, E-plane miter bend in a parallel plate waveguide illustrated in Fig. 1. The waveguide is filled with air and $k_0 = 5\ \text{rads}\cdot\text{m}^{-1}$. The Dirichlet boundary condition $E_{tang} = 1\ \text{V}\cdot\text{m}^{-1}$ is imposed on the input surface on the left edge, corresponding to the fundamental TEM mode, and $E_{tang} = 0$ is imposed on the other 6 edges of the bend, representing the metal plates of the waveguide and a terminating short circuit.

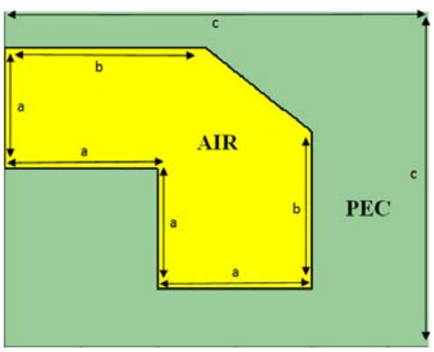

Fig. 1. Cross section of the miter bend. $a = 1$ m, $b = 1.3$ m, c $= 2.75$ m

The problem is solved in two ways. In the first method, we apply conventional CBF-NVFEM, i.e., geometric adaption only. We only refine those elements which straddle a boundary. This reduces the boundary approximation error, but field errors related to the elements which are away from boundaries cannot be reduced. In the second method, we combine geometry and field refinement by using the proposed new algorithm. Four of the meshes that arise during the adaption are shown in Fig. 2. It can be seen that the new algorithm can be started with a very coarse mesh (Fig. 2a). Notice also that it detects the field singularity around the inner corner of the bend and refines heavily there (Fig. 2d).

The error in FE functional (1) during the adaption is shown in Fig. 3 for both methods, as a function of the number of degrees of freedom (DOFs) in the mesh. The reference solution was obtained with a conventional 3D FE code using a single layer of high order tetrahedral elements and it is estimated to be acurate to 0.01 %.

It can be seen that the new method continues to reduce the error, while the conventional method stops reducing the error once the geometry is resolved sufficiently well; remaining field errors go uncorrected. Notice how the final accuracy of

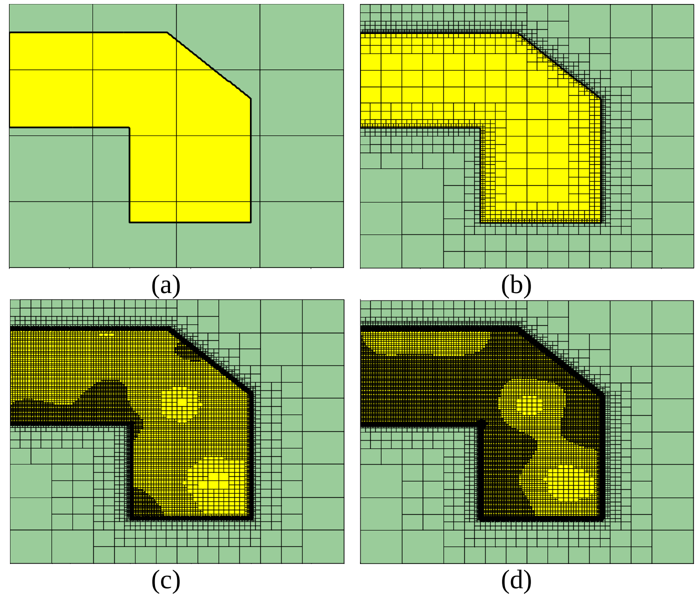

Fig. 2. Meshes obtained for the miter bend waveguide using the new algorithm.

the conventional method depends on the fineness of the initial mesh.

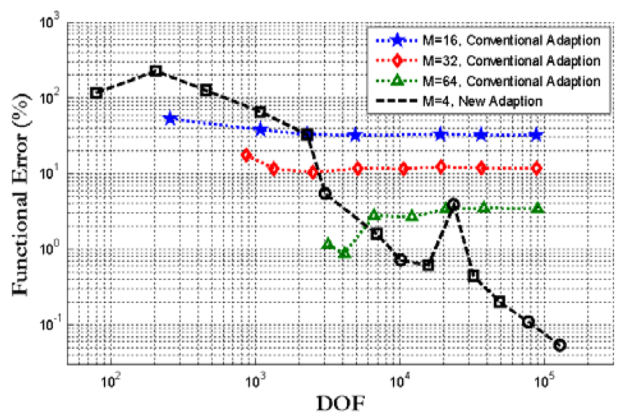

Fig. 3. Error in the FE functional versus DOFs for the miter bend test case; conventional adaptive refinement compared to the new adaptive refinement. For the "new adaption" curve, squares indicate geometric refinement (GR) and circles indicate field refinement (FR).

IV. References

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